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DEVELOPMENT OF RANDOM CHOICE NUMERICAL METHODS FOR BLAST WAVE PROBLEMS

BY HARLAND M. GLAZ

RESEARCH AND TECHNOLOGY DEPARTMENT

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Glimm's method is a relatively new numerical technique for solving hyperbolic systems of conservation laws, including those describing compressible fluid mechanics. This report contains an outline of the method, a discussion of the published numerical tests of the method, and some new numerical test results. The latter are one-dimensional models of the effects seen in blast wave propagation. The application of Glimm's method —

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to explosion problems is studied. An appendix is included to indicate how the method can be applied to problems involving a nonideal equation of state.

SUMMARY

This report presents and studies a computational method called "Glimm's method" from the point of view of applications to blast wave problems. The report includes the results of preliminary numerical studies. This work was performed by the author who is a member of the Applied Mathematics Branch of NAVSWC/WOL. Conversations with Professor Alexandre Chorin, Professor Gary Sod, Dr. Hy Sternberg, and Dr. Greg Shubin are gratefully acknowledged.
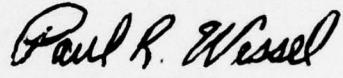
Paul R. Wessel

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INTRODUCTION

Glimm's method is a numerical technique for solving hyperbolic systems of conservation laws in one space dimension. It has been extended to problems with more than one space dimension and to systems in weak conservation form. The method does not involve finite differences; its unique features are a random choice at each time step (which implies that the solution is a random variable) and the explicit solution of a Riemann (shock-tube) problem for each pair of adjacent mesh points. The main advantage of the method is its ability to resolve discontinuities (both shock waves and contact discontinuities) exactly without smoothing. Furthermore, this takes place automatically--no special shock tracking procedures are required. In particular, the method recognizes the formation of a discontinuity and correctly resolves the interaction of discontinuities. The price that the user pays for this property is the imposition of a statistical error in the computed solution.

A precursor of Glimm's method is the finite difference scheme of Godunov. Riemann problems are solved as in the Glimm algorithm; however, there is no random choice element and the solution is advanced in time by a deterministic scheme [Reference 1]. The complete algorithm was introduced by Glimm [Reference 2]. Glimm used the algorithm as a theoretical tool to be used in obtaining existence (for all time) proofs for (weak) solutions of hyperbolic systems of conservation laws. As he defined the algorithm, the convergence rate is too slow for numerical computations; nevertheless, the paper represents a major theoretical advance. This state of affairs did not change substantially until Chorin's work was published in 1976 [Reference 3]. In this paper, Glimm's algorithm was modified in such a way that it became acceptable from the point of view of computational efficiency, and yet its theoretical properties remained unchanged.

This report will not consider the theoretical aspects of the theory of hyperbolic conservation laws, the existence and uniqueness of solutions to Riemann problems, or the convergence properties of Glimm's method applied to such conservation laws. There is an extensive literature on these topics and we refer the reader to two papers of Lax [References 4 and 5]; the later paper contains a substantial bibliography.

1. Godunov, S. K., "Finite Difference Methods for Numerical Computation of Discontinuous Solutions of the Equations of Fluid Dynamics," Mat. Sbornik 47, 1959, p. 271.
2. Glimm, J., "Solutions in the Large for Nonlinear Hyperbolic Systems of Equations," C.P.A.M. 18, 1965, p. 697.
3. Chorin, A. J., "Random Choice Solution of Hyperbolic Systems," J.C.P. 22, Dec. 1976, p. 517-533.
4. Lax, P. D., "Hyperbolic Systems of Conservation Laws II," C.P.A.M. 10, 1957, p. 537-566.
5. Lax, P. D., Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, Society for Industrial and Applied Mathematics, 1973.

Our main objective is to indicate how Glimm's method can be applied to blast wave problems. The advantages given above for Glimm's method indicate why such an application would be desirable. In a blast wave, it is the discontinuous portions of the flow field which are of the greatest importance; any method specifically designed for these regions should be quite useful. Furthermore, the statistical error which is inherent in the method will not be important in most applications. These points will again be taken up in the last section of this report.

Blast wave problems can be formulated as systems of hyperbolic conservation laws, either strong or weak depending on the geometry of the problem. We refer, of course, to the conservation laws of gas dynamics-conservation of mass, momentum, and energy. Thus, the equations of gas dynamics will be the only application of Glimm's method to be considered here in any detail. In the next section, the basic Glimm algorithm for the one-dimensional planar case is described. Our treatment is based on two papers of Chorin [References 3 and 6]. However, we include a discussion of some very simple examples to motivate the technique. The next section takes up the two-dimensional planar case and follows Chorin [Reference 3]. The equations of gas dynamics are in strong conservation form for planar symmetry. This is no longer the case for spherical, cylindrical, and axisymmetric symmetry and the extension of Glimm's method to the weak conservation form case is taken up in the following section. Throughout these three sections, the results of numerical tests will be presented; this will include some new results. In the final section, some blast wave problems will be discussed from the point of view of Glimm's method. One interesting point in such applications is that the equations of state which are of interest can be quite complicated. However, Glimm's method has thus far only been applied to a γ-law gas. We hope that this gap can be considered partially filled by the appendix to this report where we show how Riemann problems can be solved for an arbitrary equation of state.

GLIMM'S METHOD

In this section, the Glimm algorithm for solving initial value problems for strong conservation laws in one space dimension is presented. The problem to be solved is

$$\underline{u}_t + \underline{f}(\underline{u})_x = 0$$

$$\underline{u}(x,0) \text{ given}$$

(1)

where $\underline{u} = \underline{u}(x,t)$ is a function defined on the half-plane $t \geq 0$. Physically, the vector $\underline{u}$ represents the conserved quantities and the vector $\underline{f}$ represents the fluxes. The system (1) is the differential version of the integral conservation laws. The equations of one-dimensional gas dynamics with planar symmetry are

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6. Chorin, A. J., "Random Choice Methods with Applications to Reacting Gas Flow," J.C.P. 25, Nov. 1977, p. 253-272.

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

$$e_t + ((e + p)u)_x = 0 \tag{2}$$

$$e = \rho\varepsilon + \tfrac{1}{2}\rho u^2, \quad \varepsilon = F(p,\rho)$$

where ρ = density, u = velocity, e = energy per unit volume, ε = internal energy per unit mass, and p = pressure. The equation $\varepsilon = F(p,\rho)$ is the equation of state and we assume that this relation can be inverted to find the pressure p as a function of ε and ρ. If we set $\underline{u} = (\rho, \rho u, e)^T$, it is clear that the system (2) can be put in the form (1) after using the equation of state to eliminate the pressure terms. A computation shows that the system (2) is hyperbolic for any reasonable equation of state.

The method of computation for solving the system (1) is as follows. Let k be an increment of time and h a spatial increment. The solution is to be obtained at the times $t = nk$ at the points $x = ih$, $i = 0, \pm 1, \pm 2, \cdots$ and at the times $t = (n+\tfrac{1}{2})k$ at the points $x = (i+\tfrac{1}{2})h$, $i = 0, \pm 1, \pm 2, \cdots$ is here n ranges over the nonnegative integers. Let $\underline{u}_i^n \sim \underline{u}(ih, nk)$, $\underline{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}} \sim \underline{u}((i+\tfrac{1}{2})h, (n+\tfrac{1}{2})k)$. To define the algorithm, it is necessary and sufficient to describe how to find $\underline{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}}$ given $\underline{u}_i^n$, $\underline{u}_{i+1}^n$. Consider the initial value problem for the system (1) with initial data

$$\underline{u}(x,0) = \begin{cases} \underline{u}_{i+1}^n, & \text{for } x \geq 0 \\[2mm] \underline{u}_i^n, & \text{for } x < 0. \end{cases}$$

This is called a Riemann problem. Let $\underline{v}(x,t)$ denote the solution of this problem and let θ_i be a value of a random variable θ equidistributed in $[-\tfrac{1}{2}, \tfrac{1}{2}]$. Let $\underline{u} = \underline{v}(\theta_i h, k/2)$ = the value of the solution of the Riemann problem at the point $(\theta_i h, k/2)$. Set

$$\underline{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \underline{u}.$$

This procedure is repeated for each i and a similar construction takes place from $t = (n+\tfrac{1}{2})k$ to $t = (n+1)k$.

This construction is illustrated in Figures 1, 2, and 3. Figure (1a) is a schematic diagram of a shock tube at $t = 0$ with constant states $\underline{u}_\ell, \underline{u}_r$ to the left and right of a membrane located at $x = 0$. Figure (1b) shows the same shock tube at some later time $t = t^*$ after the membrane has been burst. In this illustration, a shock is moving to the right and a rarefaction is moving to the left. Figure (1c) shows the solution in the (x,t)-plane; note that the solution depends only on the ratio x/t. Of course, Figure 1 is only one example of a solution to a Riemann problem; which waves form, their velocities, and the values of the state variables in the *-state must be determined by some means. Figure 2 illustrates the situation in the (p,u)-plane. There is a one-parameter family of states (p,u) which can be connected to each of (p_ℓ, u_ℓ), (p_r, u_r)

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$\underline{U}_\ell = (\rho_\ell, U_\ell, P_\ell)$ $\underline{U}_r = (\rho_r, U_r, P_r)$ $t = 0$
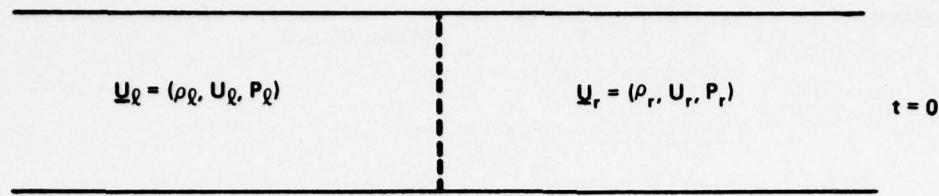
(1a) INITIAL CONFIGURATION OF THE SHOCK TUBE

$\underline{U}_\ell$ (P_*, U_*) $\rho_{*\ell}$ ρ_{*r} $\underline{U}_r$ $t = t^* > 0$
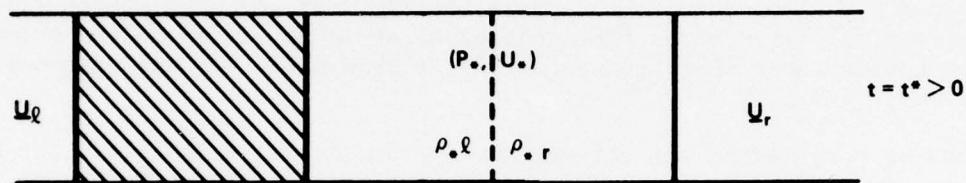
(1b) CONFIGURATION AT A LATER TIME

CONTACT DISCONTINUING

(P_*, U_*)

SHOCK FRONT

$\rho_{*\ell}$ ρ_{*r}

$\underline{U}_\ell$ $\underline{U}_r$
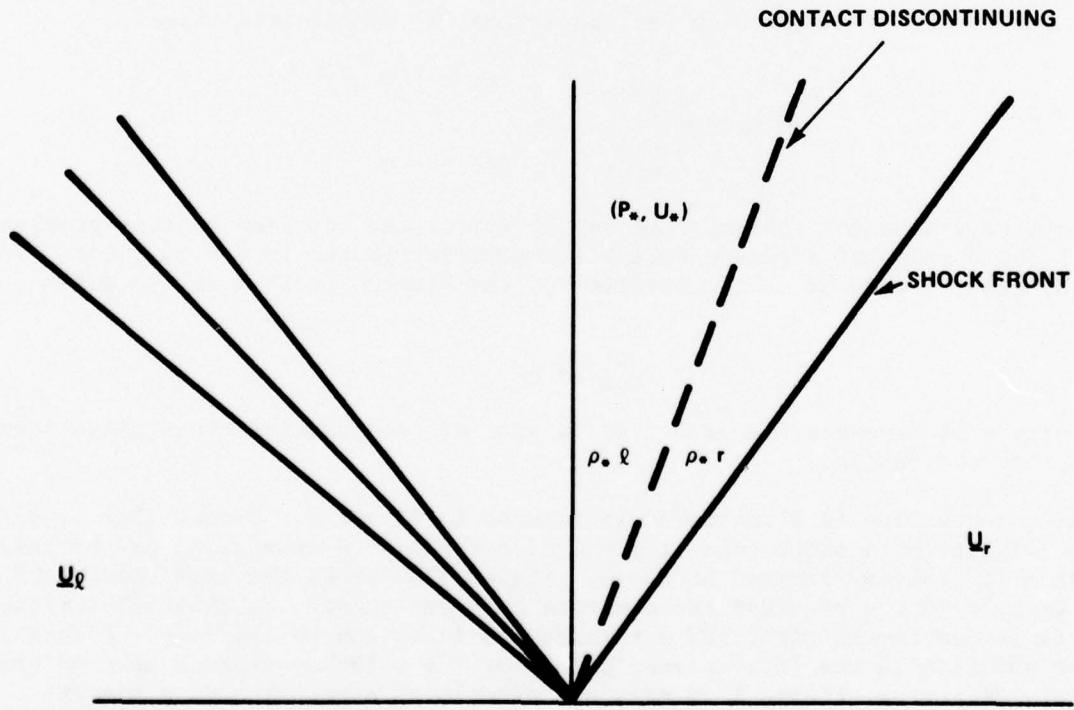
(1c) THE SOLUTION IN THE (x, t) PLANE

FIGURE 1. SOLUTION OF THE SHOCK TUBE PROBLEM FOR THE CASE OF A FORWARD SHOCK AND A BACKWARDS RAREFACTION .
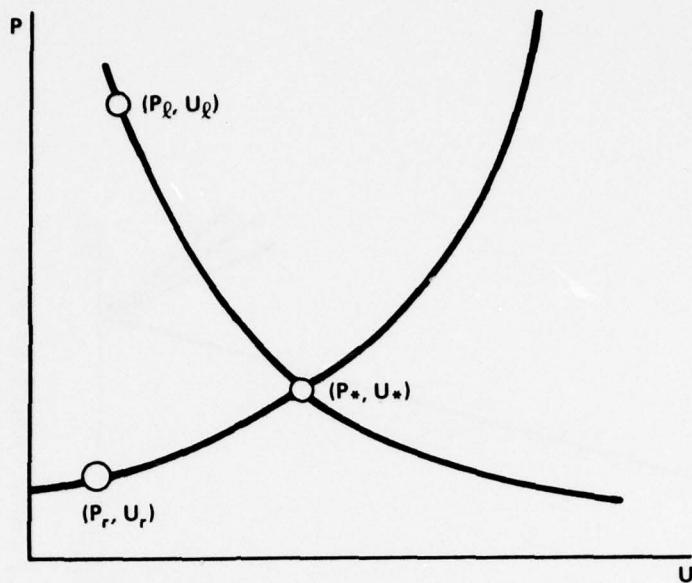
FIGURE (2a). THE CASE OF A SHOCK MOVING TO THE RIGHT AND A RAREFACTION TO THE LEFT.
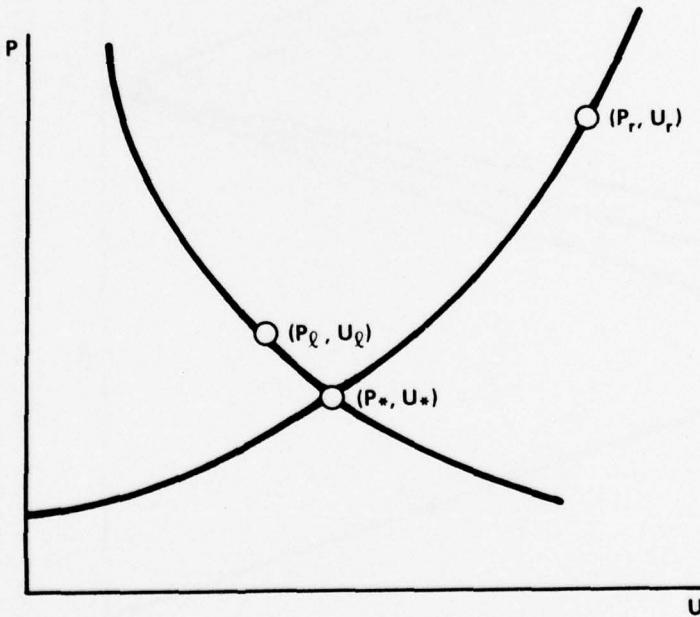
FIGURE (2b). FOR THIS CONFIGURATION, RAREFACTION WAVES MOVE BOTH TO THE RIGHT AND TO THE LEFT.

FIGURE 2. THE ITERATION IN THE PRESSURE-PARTICLE VELOCITY (p – u) PLANE.
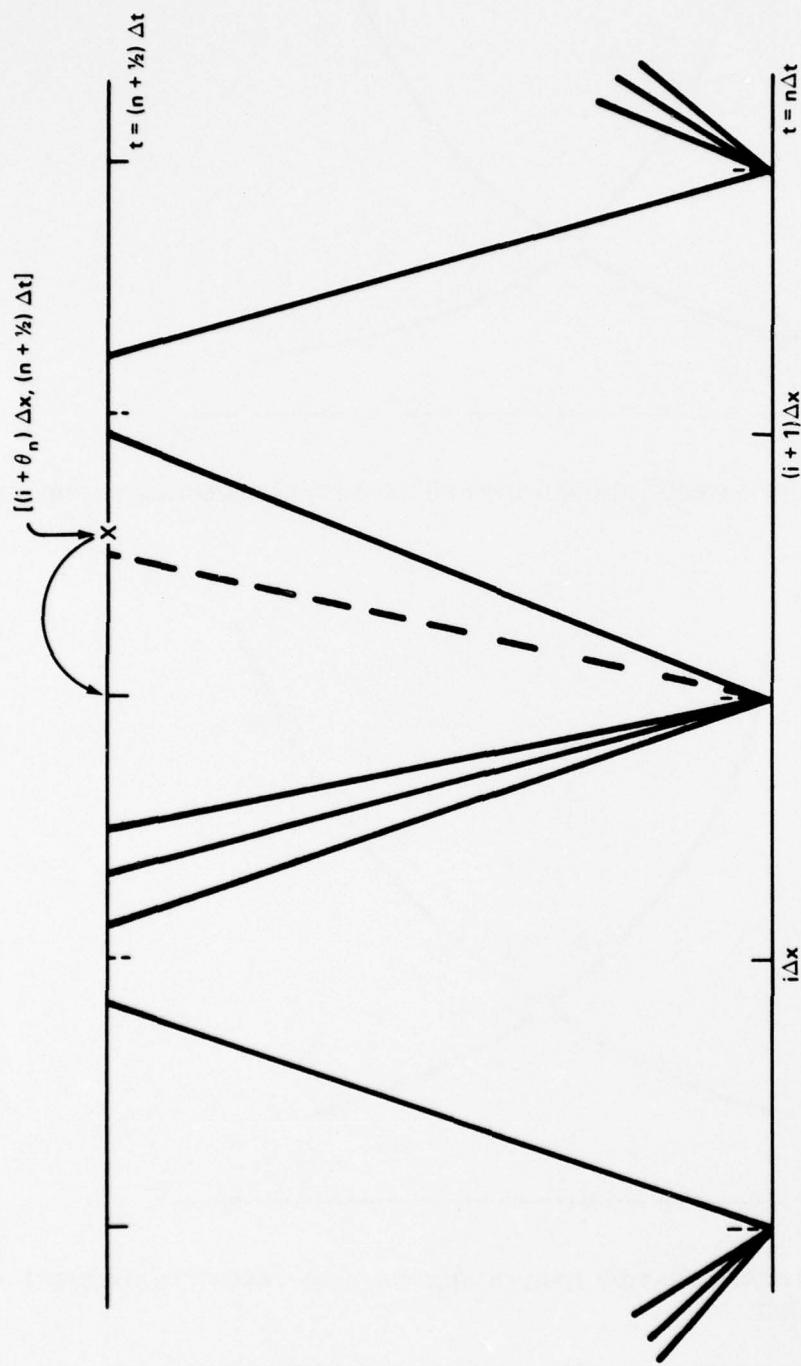
FIGURE 3. A HALF TIME STEP IN GLIMM'S METHOD. θ_n IS THE VALUE OF A RANDOM VARIABLE θ WHICH IS EQUIDISTRIBUTED ON $(-\frac{1}{2}, +\frac{1}{2})$. THE RIEMANN PROBLEM WITH $\underline{U}_\ell = \underline{U} \, (i\Delta x, \, n\Delta t)$ AND $\underline{U}_r = \underline{U}$ $((i + 2) \, \Delta x, \, n\Delta t)$ IS SOLVED AND SAMPLED; THEN $\underline{U} \, (i + \frac{1}{2})\Delta x, \, (n + \frac{1}{2})\Delta t)$ IS SET EQUAL TO THE SAMPLED VALUES.

by either a uniform shock or a simple wave (centered rarefaction fan). A general proof of this fact may be found in Reference 5. These one-parameter families are represented by the curves in Figure 2. The intersection of the two curves through (p_ℓ, u_ℓ) and (p_r, u_r) must be (p_*, u_*); once the latter is known, the complete Riemann problem solution is easy to find. The analytical method used to find (p_*, u_*) - the Godunov iteration - is explained in detail in the appendix to this report after which the complete solution is constructed. Most of the ideas involved are explained and very well motivated in Reference 7. The theory for general hyperbolic systems of conservation laws may be found in References 4 and 5.

Assume now that Riemann problems can be solved for the system (2). The mechanics of Glimm's method are illustrated in Figure 3. At time $t = n\Delta t$, the Riemann problem, setting the values of the state variables at $x = i\Delta x$ and $x = (i+1)\Delta x$ to be $\underline{u}_\ell, \underline{u}_r$, is constructed with the membrane at $x = (i+\frac{1}{2})\Delta x$. The solution to this Riemann problem is found (see the appendix) and is then sampled along the line $t = (n+\frac{1}{2})\Delta t$ where the sample point has the uniform distribution on the segment $[i\Delta x, (i+1)\Delta x]$. The values of $\underline{u}$ at the sample point are then assigned to $\underline{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}}$. This procedure is then repeated for each pair of adjacent mesh points along the x-axis; then a new random number is selected and one proceeds to the next time step. For a more thorough discussion of the algorithm and its properties, see References 3 and 6.

Glimm's method is unconditionally stable. Nevertheless, the Courant condition must be satisfied; if not, waves will leave the sampling interval which implies that the sampling probabilities will be incorrect. In effect, this changes the problem being solved. The success of the method is dependent on the choice of random numbers $\{\theta_n\}$. Chorin introduced two new features here--first, only one value of θ for each time step (instead of choosing a new random number for each mesh point every time step as Glimm did in his paper) is used and, second, special techniques are used to insure that the sequence approaches rapidly the uniform distribution. A recent result of Liu [Reference 8] has shown that it is not necessary to use random numbers at all; all that is necessary is that the sequence used approach equidistribution. Thus, in the near future, it will be possible to use predetermined sequences which are chosen to optimize the approach to equidistribution; this development will improve the solutions obtained with Glimm's method.

Glimm's method is only first-order accurate but has infinite resolution in a sense to be described by a few simple examples. In Figure (4a), we have chosen two constant states $\underline{u}_\ell$ and $\underline{u}_r$ in such a way that the solution to the associated Riemann problem is a unit speed shock moving to the right (with no wave at all moving to the left and no contact discontinuity). This is an easy case to analyze.

---

7. Courant, R. B., Friedrichs, K. O., <u>Supersonic Flow and Shock Waves</u>, Wiley-Interscience, 1948.

8. Liu, T. P., "The Deterministic Version of the Glimm Scheme," <u>Comm. Math. Phys.</u> <u>57</u>, 1977, p. 135-148.
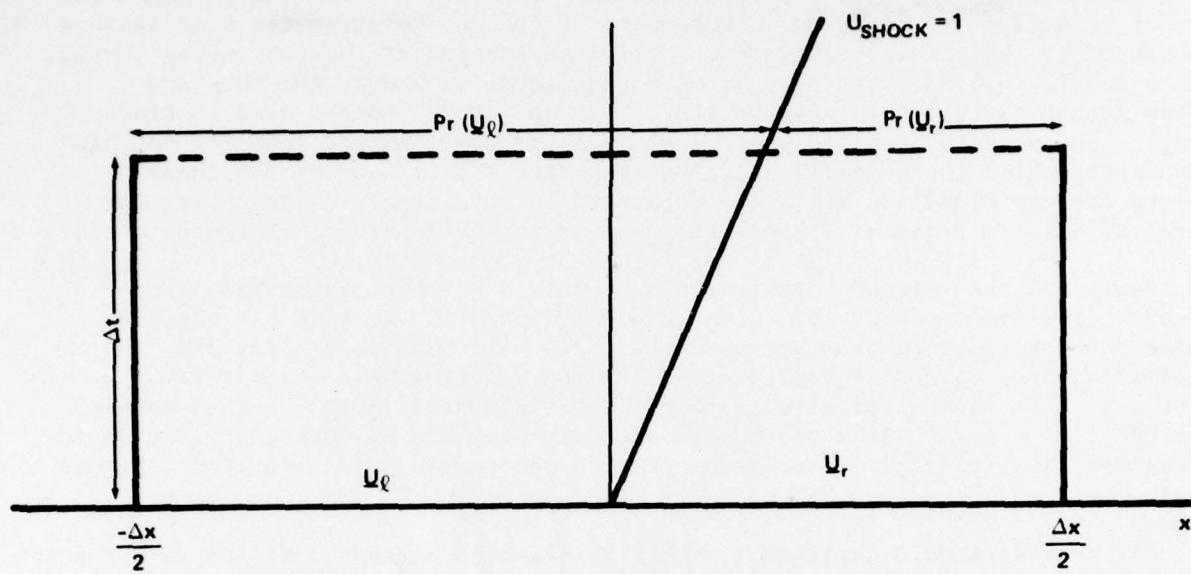
(4a) THE CONFIGURATION FOR A REIMANN PROBLEM WHOSE SOLUTION IS A SHOCK:
THIS IS A LIMITING CASE IN WHICH THE LEFT WAVE AND CONTACT DISCONTINUITY
ARE NOT PRESENT.

(4b) LEFT, LEFT CHOICE

(4c) LEFT, RIGHT CHOICE

(4d) RIGHT, LEFT CHOICE

(4e) RIGHT, RIGHT CHOICE
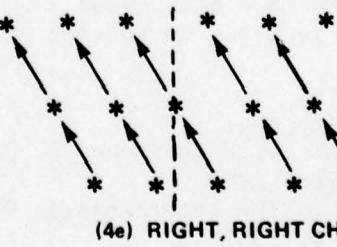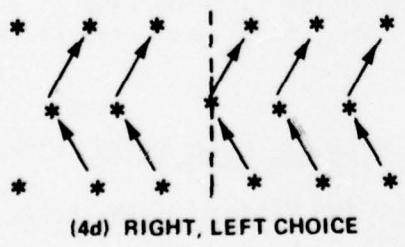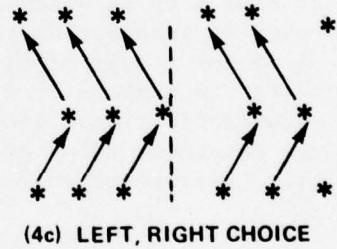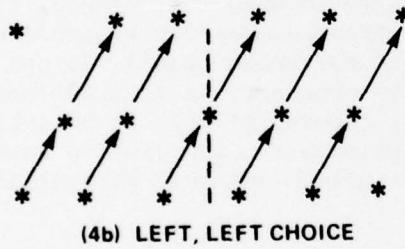
FIGURE 4. THE SOLUTION FOR A UNIT SPEED UNIFORM SHOCK WAVE USING GLIMM'S METHOD.

When choosing the random number in Glimm's method, there are only two possibilities --either $\underline{u}_\ell$ is obtained or $\underline{u}_r$ is obtained. These occur precisely when the random choice lies to the left or the right of the shock, respectively. For two consecutive half-time steps, there are four possibilities and these are illustrated in Figures (4b) - (4d). For example, Figure (4b) illustrates the fact that if two consecutive random numbers are chosen to the left of the shock, then the shock moves forward one mesh point. From Figure (4a) it is clear that $pr(\underline{u}_\ell) \equiv$ probability that the random number lies to the left of the shock $= \frac{1}{2} + \frac{\Delta t}{2\Delta x}$ and $pr(\underline{u}_r) = \frac{1}{2} - \frac{\Delta t}{2\Delta x}$. Hence,

$$pr(\underline{u}_\ell)^2 - pr(\underline{u}_r)^2 = \frac{\Delta t}{\Delta x}.$$

This says that the net probability of the shock advancing one mesh point per full time step is $\Delta t/\Delta x$. The velocity inherent in an advance of one mesh point is just $\Delta x/\Delta t$. Therefore, the

$$\text{mean shock speed} = (\Delta t/\Delta x)\cdot(\Delta x/\Delta t) = 1.$$

Hence, on the average the shock moves at the correct speed. At least as important as this result is the fact that the shock remains a shock without smearing and the values of the flow field on either side of the shock remain exact (up to the tolerance used in solving the Riemann problems). I.e., the resolution of the method is infinite for this example. Even though the location of the shock will generally be in error, one will always see a shock, it will be the correct shock, and no new unwanted waves of any kind will be introduced.

This argument involving the shock and rarefaction curves in the (p,u)-plane can be generalized to show that any Riemann problem will be solved by Glimm's method with infinite resolution despite inherent statistical errors. That is, the information content of the waves is never lost.

A somewhat more difficult problem is the interaction of the waves from two adjacent Riemann problems. The setup and analytical solution of a problem of this type is illustrated in Figure 5. The various waves remain uniform throughout the solution. We have also solved this problem using Glimm's method (for the treatment of the boundaries, which is straightforward here, see Reference 3). As in the previous problem, the wave speeds are subject to statistical error and yet the constant states are computed exactly to the precision of the Riemann problem solver. The interaction zone just below state E contained only a few mesh points; still state E was computed exactly. This illustrates, in this more complicated case, the infinite resolution of the algorithm and the property that it does not lose information.

Of course, if one is faced with arbitrary initial data, the solution will rarely consist of uniform waves. In view of the fact that the basic idea of Glimm's method is to approximate the flow field by a collection of simple waves, the question arises as to how useful the method will be for strong nonuniform waves. Before proceeding to a numerical experiment, we comment that as the mesh spacing becomes smaller, the approximation of the flow field by piecewise constant data, hence the simple wave approximation to the solution, becomes better. Thus, we expect more accurate solutions with more mesh points--just as with any finite

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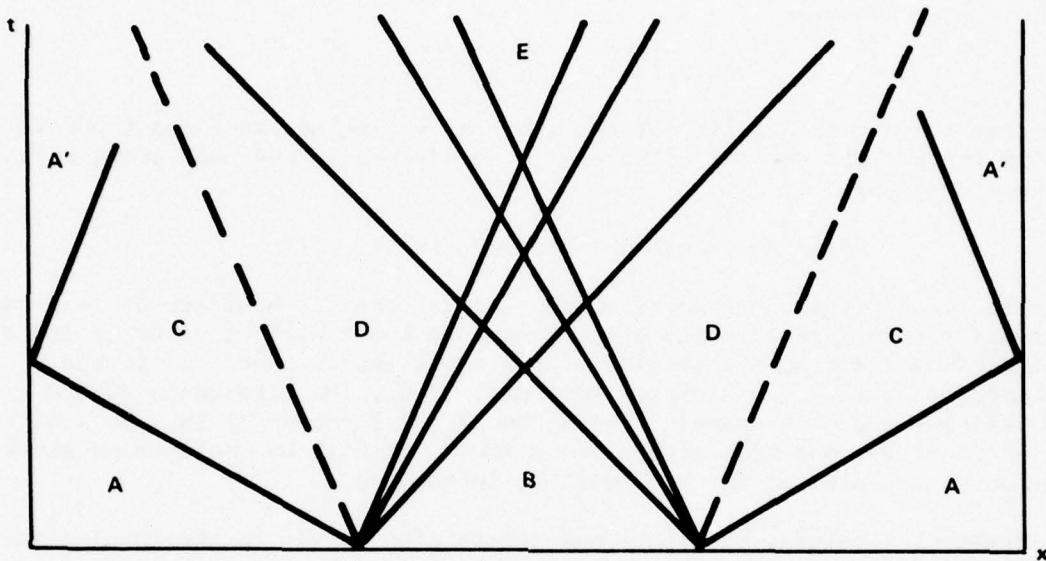
FIGURE 5. AN EXAMPLE OF THE INTERACTION OF UNIFORM WAVES. THE CONS-
TANT STATES A AND B ARE THE INITIAL DATA; THE PICTURE IS SYM-
METRIC ABOUT THE CENTER OF STATE B. THE STATES C AND D ARISE
FROM THE SOLUTION OF THE RIEMANN PROBLEM GIVEN BY A AND B.
A' IS A CONSTANT STATE GIVEN BY THE SHOCK REFLECTION EQUA-
TIONS. STATE E, WHICH ARISES FROM THE INTERACTION OF THE
RAREFACTION FANS, IS ALSO A CONSTANT STATE. FOR AN IDEAL GAS
WITH $\gamma = 3$, THE SOLUTION MAY BE OBTAINED ANALYTICALLY.

difference scheme. Still, this was not the case in the above simple examples where no information was lost so long as there were as few as two or three mesh points for each distinct wave present in the solution. The question of convergence and convergence rates for Glimm's method in such problems are unanswered; neither Glimm's original proof nor any of the later theoretical work on the method apply to the case of gas dynamics with general initial data. Thus, we pursue a numerical example.

Fortunately, exact analytical solutions are available for purposes of testing the method. We have pursued an example of an exponential shock path pushed by a piston due to Sternberg. The wave pattern between the shock and the piston is about as nonuniform as possible. The exact solution may be found in Reference 9. A comparison of this exact solution with a computation using Glimm's method is illustrated in Figure 6. Observe that both the piston and the shock have moved correctly and that the wave profile is correct. We hope to do further work with similarity solutions along these lines, especially in the reactive flow case (see Reference 6 for an extension of Glimm's method to this case; also, the operator splitting idea to be presented in a later section can be applied). Two papers of Sternberg [References 9 and 10] will be the basis for these computations.

This completes our survey of the basic Glimm's method for strong hyperbolic conservation laws in one space dimension. As a final remark, we note that both the theory and the practice of the method are easier to understand for single conservation laws than for systems. Burger's equation, $u_t + uu_x = 0$ is a good example. The situation considered in Reference 11 is especially interesting in this regard.

GLIMM'S METHOD - EXTENSION TO TWO INDEPENDENT VARIABLES. The extension of Glimm's method to systems of strong conservation laws in more than one space dimension would be straightforward if multidimensional Riemann problems could be solved. For example, the two-dimensional Riemann problem consists of the specification of four constant states, one for each quadrant of the plane, at $t = 0$; and the solution to this problem would consist of using this configuration as the initial state and finding the complete solution of the equations in a neighborhood of the singularity at the origin for all $t > 0$. Unfortunately, very little is known about this problem. Also, if an exact solution (or iteration procedure) is ever discovered, it is bound to be considerably more difficult than the one-dimensional case.

The procedure developed by Chorin [Reference 3] is to use the one-dimensional Glimm algorithm as a building block in a fractional step method. In place of the two half-steps of the one-dimensional method, one takes four quarter steps for the two-dimensional method; each quarter step is a sweep in either the x or y direction.

9. Sternberg, H. M., "Similarity Solutions for Reactive Shock Waves," Quart. J. of Mech. and Appl. Math. 23, Feb. 1970, p. 77-99.
10. Sternberg, H. M., "Constant Velocity Reactive Shock Waves for Testing Numerical Methods," preprint, 1978.
11. Concus, P. and Proskurowski, W., "Numerical Solution of a Nonlinear Hyperbolic Equation by the Random Choice Method," Lawrence Berkeley Laboratory Report LBL-6487 Rev., Dec. 1977.

EXAMPLE: NONUNIFORM PLANE STRONG SHOCK WAVE

PISTON PATH
SHOCK PATH
$U = \exp(Kt_2)$
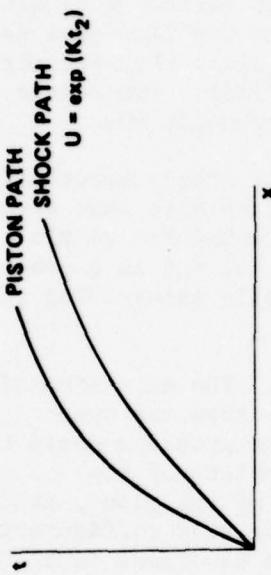
THIS IS A SIMILARITY SOLUTION. THE EXACT SOLUTION, UP TO A
NUMERICAL QUADRATURE, IS EASILY OBTAINED. SEE H. M. STERNBERG,
"SIMILARITY SOLUTIONS FOR REACTIVE SHOCK WAVES," QUART. J. MECH.
AND APPL. MATH. 23, FEB. 1970.

= EXACT SOLUTION
------- = GLIMM COMPUTATION
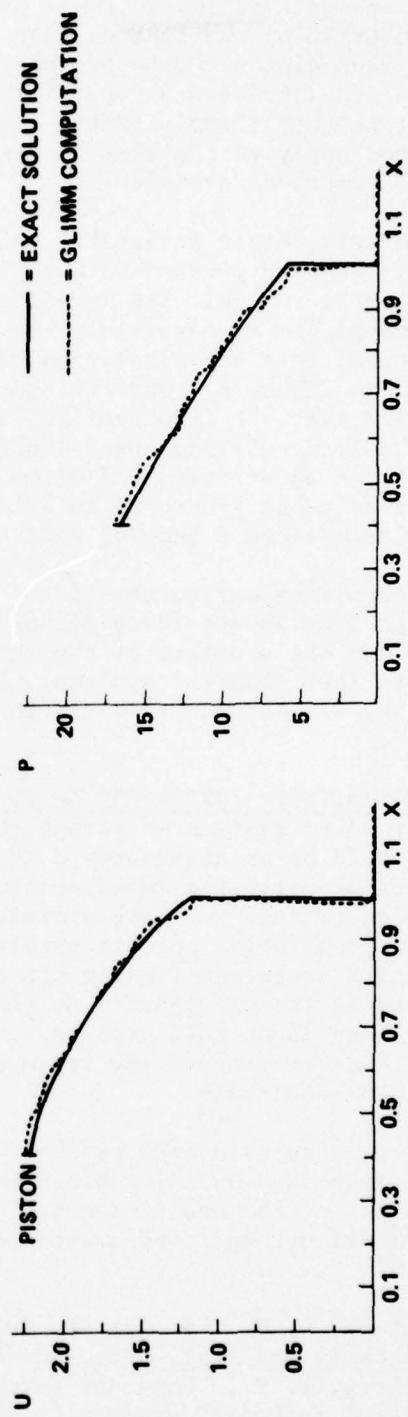
FIGURE 6. COMPUTED AND EXACT VELOCITY AND
DENSITY PROFILES FOR THE
EXPONENTIAL SHOCK PATH SIMILARITY
SOLUTION.

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This idea can be generalized to more dimensions, but the discussion here will be restricted to the two-dimensional case. The equations of two-dimensional compressible flow with planar symmetry are

$$\rho_t + (\rho u)_x + (\rho v)_y = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0$$
$$(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0 \tag{3}$$
$$e_t + ((e + p)u)_x + ((e + p)v)_y = 0$$
$$e = \rho\epsilon + \tfrac{1}{2}\rho(u^2 + v^2), \quad \epsilon = F(p,\rho).$$

This is clearly of the form

$$\underline{u}_t + \underline{f}(\underline{u})_x + \underline{g}(\underline{u})_y = 0$$

where the conserved quantities are $\underline{u} = (\rho, \rho u, \rho v, e)^T$. As before, it is assumed that the equation of state may be inverted to find p as a function of ϵ and ρ, hence as a function of the conserved quantities. In this manner the flux vectors $\underline{f}$ and $\underline{g}$ may be found from (3). The equations to be solved in an x-sweep are

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$(\rho v)_t + (\rho uv)_x = 0 \tag{4}$$
$$e_t + ((e + p)u)_x = 0.$$

That is, all y-derivatives are set equal to zero. The third of these equations becomes

$$v_t + (uv)_x = 0 \tag{5}$$

after application of the first equation. This means that the y-component of velocity v is transported as a passive scalar in an x-sweep. This is a particularly simple situation for Glimm's method; the solution of a Riemann problem for the 1st, 2nd, and 4th of equation (4) is independent of equation (5). Hence, an x-sweep reduces to the system (2) of one-dimensional flow. In view of the third of equations (4), v is conserved in the mean which implies that

$$e = \rho\epsilon + \tfrac{1}{2}\rho u^2 + k \tag{6}$$

where k is a constant. The constant plays no role and we are indeed in the situation of system (2). Equations similar to (4), (5), and (6) hold in a y-sweep.

Two questions are fairly obvious concerning this procedure. Is the two-dimensional Riemann problem consistently approximated by the fractional step procedure? Whether the answer to the first question is yes or no, does the

technique converge to the solution of the general initial value problem, assuming that the one-dimensional Glimm method does so? Unfortunately, the answers to these questions are not known. The best that one can do is to obtain numerical results for the technique. In the case of smooth solutions, a general theory is available concerning the accuracy of operator splitting [Reference 12].

There are only two such tests available at present in the literature. Concus and Proskurowski [Reference 11] consider the case of a single conservation law; their initial conditions are somewhat specialized. We shall not take up their results here except to note that they are good. The system (3) has been solved by Chorin [Reference 3] for the case of a channel with a ramp. Figure (7a) shows the geometry and the initial conditions (uniform flow). It is assumed that the gas is polytropic. There are two adjustable parameters, M = Mach number and γ = ratio of specific heats. Two cases are studied for which an analytical steady state solution is known to exist. These analytical solutions are illustrated in Figures (7b) and (7c). There is a qualitative bifurcation in the solution from a regular to a Mach reflection pattern as the parameters are varied. The results for this problem reported by Chorin arise from using Glimm's method for the equations of unsteady gas dynamics (i.e., system (3)) and waiting for the approach to the steady state. The results are very encouraging; the qualitative features are correct in both cases, quantitative results are very close, and the number of mesh points is small.

GLIMM'S METHOD - EXTENSION TO EQUATIONS IN WEAK CONSERVATION FORM. In the case of one space dimension, the problem to be solved is

$$\underline{u}_t + \underline{f}(\underline{u})_r = -\underline{w}(\underline{u})$$

$$\underline{u}(r,0) \quad \text{given.}$$

(7)

As before, $\underline{u}$ represents the vector of "conserved" quantities. The main situation of interest is the equations of one-dimensional gas dynamics with spherical or cylindrical symmetry. Explicitly, we have

$$\rho_t + (\rho u)_r = - (\eta - 1)\rho u/r$$

$$(\rho u)_t + (\rho u^2 + p)_r = - (\eta - 1)\rho u^2/r$$

$$e_t + ((e + p)u)_r = - (\eta - 1)(e + p)u/r$$

$$e = \rho\varepsilon + \tfrac{1}{2}\rho u^2, \quad \varepsilon = F(p,\rho)$$

where r is the radial coordinate, u is the velocity in the radial direction, and η is a constant which is equal to 2 for cylindrical symmetry and 3 for spherical symmetry. The remainder of the notation is as before. The new feature here is the appearance of the inhomogeneous term $\underline{w}(\underline{u})$; it is not possible to put the system (8) in strong conservation form. Another new problem is that $\underline{w}(\underline{u})$ depends explicitly on r; for problems involving significant phenomena near the origin, the equations become singular.

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12. Gottlieb, D., "Strang-Type Difference Schemes for Multidimensional Problem," Siam J. Num. Anal. 9, Dec. 1972, p. 650 & 661.

$$P = \rho = 1, v = 0$$
$$u = M(\gamma P/\rho)^{1/2}$$

(7a) CONFIGURATION FOR REGULAR TO MACH REFLECTION TRANSITION, $\alpha = \tan^{-1}(1/5)$
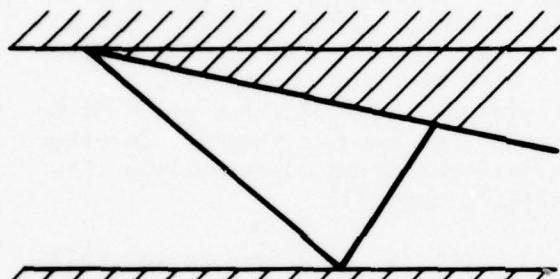
(7b) THE REGULAR REFLECTION SOLUTION
FOR M = 2.0 and γ = 1.4
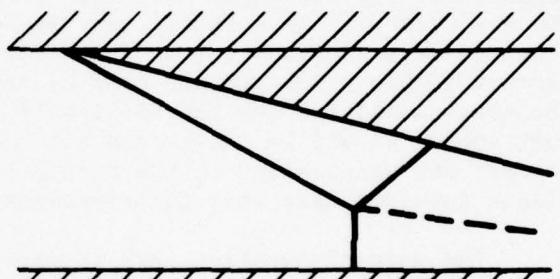
(7c) THE MACH REFLECTION SOLUTION FOR
M = 1.6 and γ = 1.2

FIGURE 7. TRANSITION FROM REGULAR TO MACH REFLECTION. THE BOTTOM TWO DRAWINGS SHOW THE
FLOW PATTERNS RESULTING FROM ANALYTIC SOLUTIONS WITH M, γ AS INDICATED.

Application of Glimm's method to the system (7) involves the notion of operator splitting. In the general case, given a system $u_t = A(u) + B(u)$, one may consider solving $u_t = A(u)$ and $u_t = B(u)$ in sequence; as the time step approaches zero, one hopes that the solution of the latter problem approaches the solution of the original problem. In special cases (usually linear operators) this can be proved, for smooth solutions, using functional analysis (product formulas), or a simple error analysis [Reference 12]. The reason for using operator splitting is that one can solve $u_t = A(u)$ and $u_t = B(u)$ separately but not the original equation. This remark applies to the system (7); consider the splitting

$$u_t + f(u)_r = 0$$

$$u_t = - w(u)$$

(9)

of (7). The first half of (9) is now in strong conservation form and the basic Glimm's method applies to its numerical solution. The second half of (9) is a system of ordinary differential equations, one for the value of each conserved quantity at each mesh point. Observe that conserved quantities at different mesh points are uncoupled in this system; however, for the case of gas dynamics - equations (8), the three conserved quantities at a given mesh point are coupled. In any event, it is a fairly easy system to solve (numerically). The $1/r$ term is treated as a constant given by $r = i\Delta x$ for the ith mesh point (other alternatives are available which are not treated here). Note also that the origin is a boundary condition for the first half of (9); therefore, equations for $u(x = 0)$ do not appear at all in the second half of (9) because they are not needed. In other words, the singularity at the origin has disappeared (at least numerically). The ideas involved here were first presented by Sod [Reference 13].

The natural question here is whether the splitting (9) converges to the full operator (7) as the time step approaches zero, under the assumption that the Glimm operator itself converges (of course, there is no question about the convergence of numerical approximations to the system of ordinary differential equations). There are some positive results for the case of a single conservation law (private communication with G. Sod). However, little is known for the case of gas dynamics, equations (8).

An alternative to Sod's splitting procedure would be to solve the Riemann problem for the full system (7). This is substantially more difficult than the corresponding planar problem, but it appears that a mathematical analysis would yield results—this is not clear for the two-dimensional Riemann problem discussed in the previous section. Indeed, through the use of power series expansions, the Riemann problem for the system (8) of gas dynamics with spherical symmetry ($\eta = 3$) has been solved approximately in the strong shock case (i.e., the ratio p_ℓ/p_r is large). Details may be found in References 14 and 15. Figure 8 is a

13. Sod, G. A., "A Numerical Study of a Converging Cylindrical Shock," J.F.M. 83, 1977, p. 785-794.
14. Friedman, M. P., "A Simplified Analysis of Spherical and Cylindrical Blast Waves," J.F.M. 11, 1962, p. 1-15.
15. Holt, M., "The Initial Behavior of a Spherical Explosion. I. Theoretical Analysis," Proc. of the Royal Society, A, 234, 1956, p. 89-109.
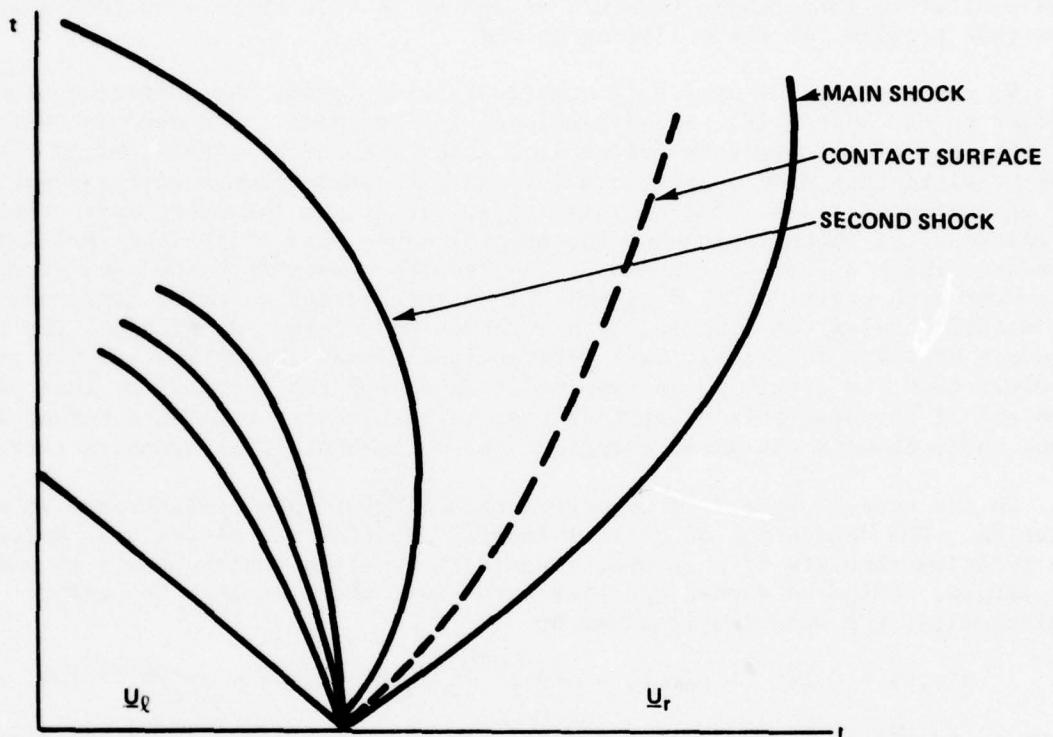
FIGURE 8.   A SCHEMATIC ILLUSTRATION OF THE GENERAL SOLUTION TO THE ONE-DIMENSIONAL RIEMANN PROBLEM IN SPHERICAL GEOMETRY.

schematic illustration of the solution in the strong shock (moving away from the origin) case. One sees immediately that the solution is no longer simply a function of r/t, but depends on r,t separately. This makes the application of this solution to Glimm's method awkward and inefficient. Also, for such an application to be considered, it will be necessary to extend the analysis to the entire spectrum of pressure ratios; however, $p_\ell/p_r \gg 1$ is probably the most difficult case. Hence, for the time being, the splitting algorithm is the only way to apply Glimm's method to systems in weak conservation form. Even if techniques involving exact or approximate solutions of the full Riemann problem (e.g., Figure 8) prove infeasible, it would be useful to have these solutions available as a test problem for the splitting method.

We now present the available numerical evidence for the splitting algorithms applied to the system (8), one-dimensional gas dynamics. Sod studies [Reference 13] the case of a converging cylindrical shock induced by a cylindrical shock tube problem; this problem has an extensive literature from both the computational and experimental sides. The singularity at the origin (pressure approaches infinity as the shock approaches the origin) makes this a difficult problem for standard finite difference schemes. The results presented by Sod are in good agreement with experimental data and are an improvement on prior computations. The method handles the singularity accurately and without difficulty, the correct sequence of waves is formed; wave interactions appear correctly, and the reflection of waves from the origin is as expected. In short, the results for this problem have all of the desirable properties that we have seen for Glimm's method in the plane case, despite the added complication of the nontrivial symmetry terms.

In the case of spherical symmetry, the Primakoff problem provides an exact solution. The derivation of this similarity solution may be found in Reference 7. The solution consists of a curved, nonuniform waveform, which decays to zero at the origin, behind an expanding shock wave front which is sharply peaked. Analytically, the solution is given by

$$u(r,t) = 0.1rt^{-1}; \quad p(r,t) = \rho_o r^3 t^{-12/5}/25k; \quad \rho(r,t) = 4\rho_o rt^{-2/5}/3k \qquad (10)$$

where ρ_o is the constant ambient density and k is a constant. The location of the shock is given by

$$R(t) = kt^{2/5}. \qquad (11)$$

We chose (arbitrarily) t = 1 for the initial time in our computations and used the discretized versions of equations (10) and (11) for initial conditions. At t = 5.6, R(t) is roughly double R(1) and so this was chosen as the final integration time. Figures 9 and 10 reproduce density and pressure profiles, respectively, at t = 5.6, for both the exact and numerical solution. Figure 11 is the peak pressure vs. distance profile for the wave from t = 1 to t = 5.6; this is an important function in evaluating blast wave effects. It can be seen that the numerical results for Glimm's method are in agreement with the exact solution; in the run represented here the shock location is exact, but in other runs it is off a few mesh points. It is not clear at this point whether this error is caused by the random choice element or by the operator splitting; in any event, we can report that reducing the mesh will reduce the error in shock location indicating that convergence is obtained for this problem. Using a new

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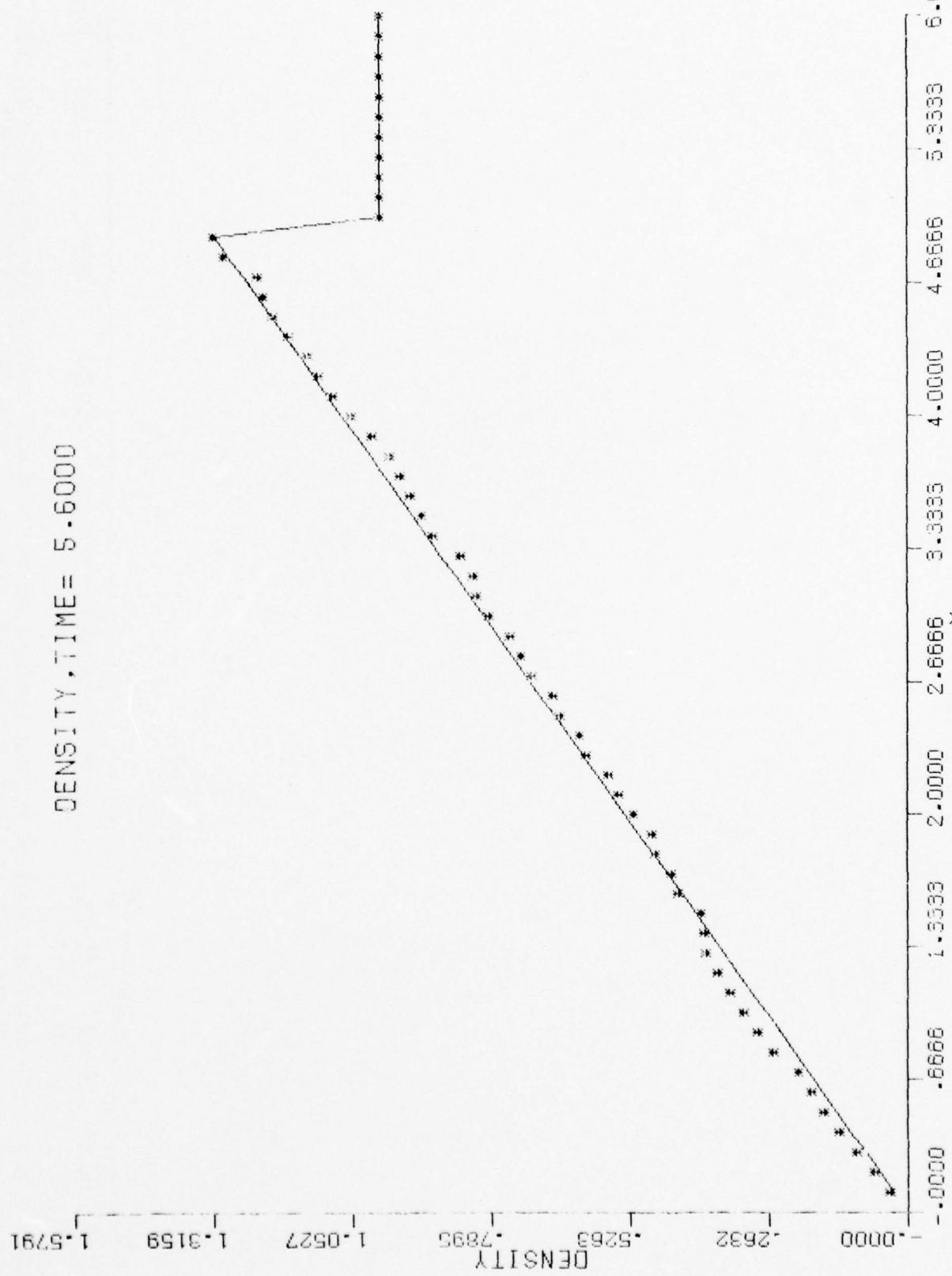
FIGURE 9. EXACT AND COMPUTED DENSITY PROFILES FOR THE PRIMAKOFF SOLUTION (SEE EQUATIONS 10). THE SOLID LINE IS THE EXACT SOLUTION; THE SLOPE OF THE SHOCK FRONT IS A PLOTTER ERROR.
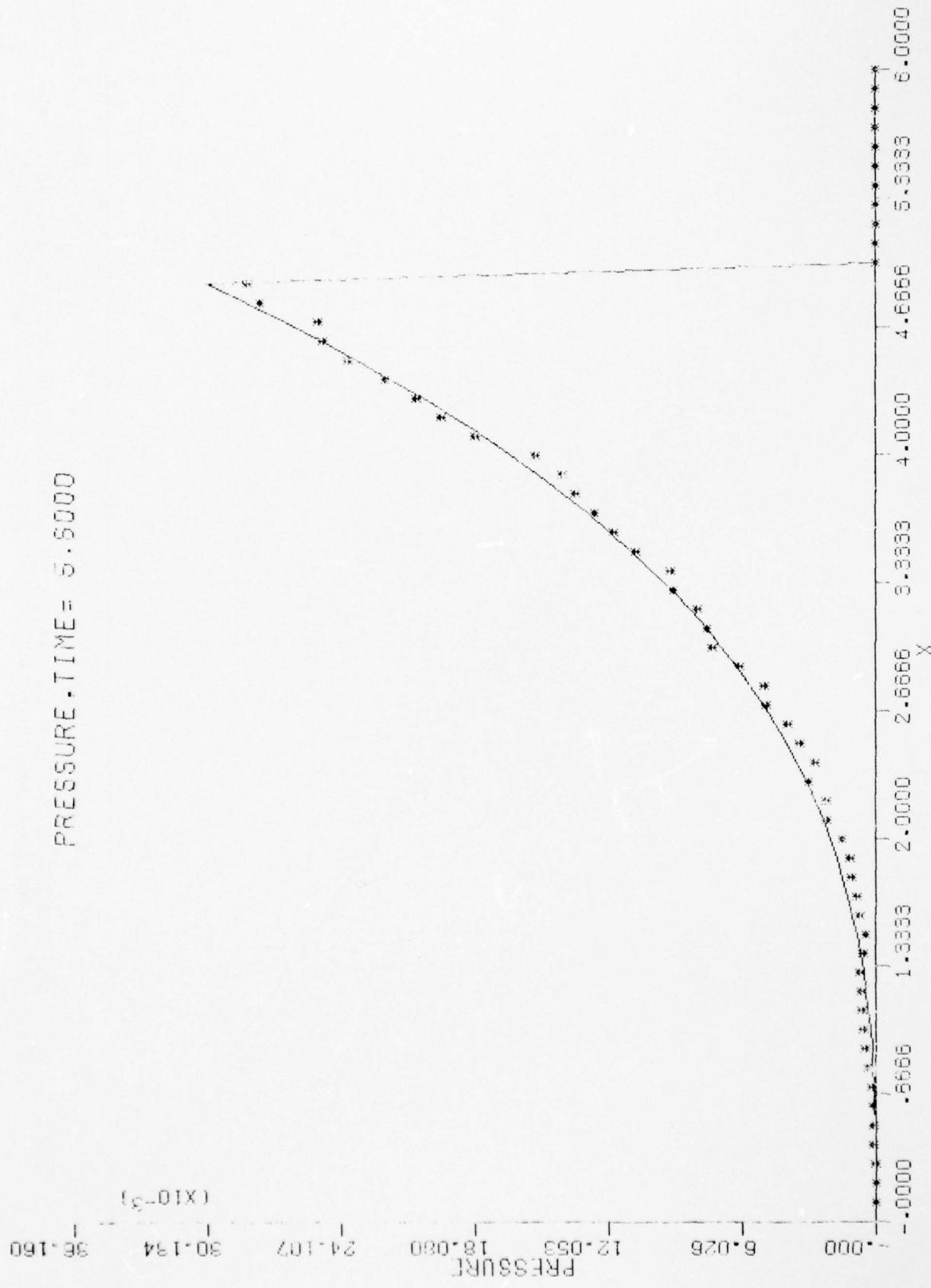
FIGURE 10. EXACT AND COMPUTED PRESSURE PROFILES FOR THE PRIMAKOFF SOLUTION (SEE EQUATIONS 10). THE SOLID LINE IS THE EXACT SOLUTION; THE SLOPE OF THE SHOCK FRONT IS A PLOTTER ERROR.
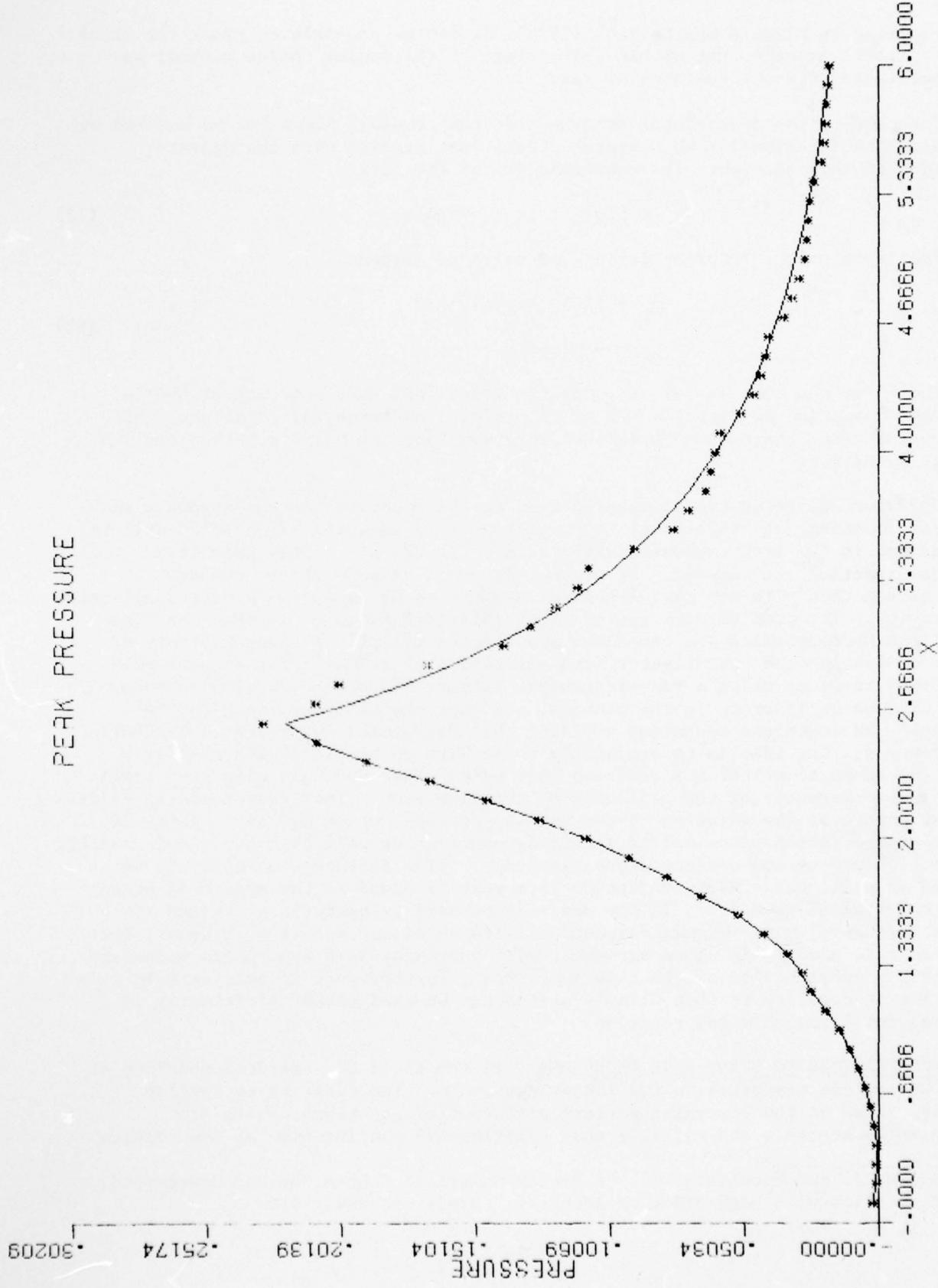
FIGURE 11. PEAK PRESSURE VS. DISTANCE PLOT FOR THE PRIMAKOFF SOLUTION (SEE EQUATIONS 10). THE SOLID LINE IS THE EXACT SOLUTION; THE PEAK AT X=2.5 HAS BEEN CLIPPED BY THE PLOTTER AND THE FIT IN THAT REGION IS BETTER THAN IT APPEARS HERE.

idea reported in Glimm & Marchesin[16] (1978), it may be possible to track the shock front (almost) exactly, yet within the context of the random choice method; we have not tested these techniques as yet.

The case of two dimensional axisymmetric compressible flows can be handled by combining the fractional step approach of the last section with the operator splitting of this section. The equations are of the form

$$\underline{u}_t + \underline{f}(\underline{u})_r + \underline{g}(\underline{u})_z = -\underline{w}(\underline{u}). \tag{12}$$

The idea is to use operator splitting and solve in sequence

$$\underline{u}_t + \underline{f}(\underline{u})_r + \underline{g}(\underline{u})_z = 0$$

$$\underline{u}_t = -\underline{w}(\underline{u}). \tag{13}$$

The first equation is solved using the fractional step approach of Chorin; the second equation is simply a system of ordinary differential equations. At the present time, there is no published work available on Glimm's method applied to this situation.

Operator splitting can also be applied to the equations of gas dynamics with chemical reaction. In this case, the vector $\underline{w}(\underline{u})$ in equation (7) or (12) will be zero except in the last component where a term for the net energy gain from chemical reaction will appear. Of course, it would be possible to combine chemical reaction with nontrivial spatial symmetries by including a more complicated vector $\underline{w}(\underline{u})$. The problem with the operator splitting approach is that the time scales for hydrodynamics and chemistry are (typically) off by several orders of magnitude leaving the investigator with the choice of either using an extremely small time step, or using a very inaccurate integration of the chemical phenomena. (This dilemma is inherent in the problem, not just the Glimm approach to the problem.) An ingenious technique avoiding this difficulty is presented by Chorin [Reference 6]; the idea is to explicitly solve Riemann problems with chemistry using the assumption that any reaction that takes place during a time step takes place instantaneously at the beginning of the time step. This preserves the self-similar nature of the solution to the Riemann problem, which now may consist of Chapman-Jouguet detonations and/or strong detonations as well as hydrodynamic shocks, rarefaction waves, and contact discontinuities. (The deflagration case can be handled as well, but a heat conduction term must be added to the system of equations on physical grounds.) In the paper, numerical evidence is presented for both strong and Chapman-Jouguet detonations with excellent results. However, the reaction rate used is in close agreement with the unphysical hypothesis necessary in Chorin's construction of the solution. Thus, further work is necessary in this area, but it does appear that Glimm's method can be used rather efficiently in problems involving chemical reaction.

APPLICATIONS TO BLAST WAVE PHENOMENA. In the field of numerical modeling of blast waves, one may perceive two long-range goals. The first is to develop methods, based on the governing partial differential equations, which are sufficiently accurate and reliable that experimental confirmation of the results

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16. Glimm, J. and Marchesin, D., "A Random Numerical Scheme for One Dimensional Fluid Flow with High Order of Accuracy," preprint, May, 1978

is not required; the achievement of this goal has obvious practical importance. The second goal is to develop the methods to such an extent that they are useful in a theoretical analysis of blast wave phenomena--a field of study which is by no means completely understood. These two goals are clearly interrelated; however, the exact relationship is not obvious at this time.

The difficulties in solving blast wave problems can usually be traced to the existence of discontinuities in the flow field--this includes shock waves, slip surfaces, and material interfaces. The calculation in a neighborhood of the discontinuities is both the most difficult part to solve numerically and the most important part from a practical point of view. Indeed, the solution is almost useless unless it contains peak pressure vs. distance plots, shock trajectories, interface trajectories, etc. Furthermore, a typical phenomenon in blast wave computations is the formation, interaction, and disappearance of discontinuities in the midst of the computation; this includes bifurcations of discontinuity patterns such as the transition from regular to Mach reflection. The physical effects of this behavior are critical in the applications.

The development of numerical techniques to handle these problems began in World War II and continues today. Reference 17 contains a recent review of several of these methods. Most rely on artificial viscosity and "shock capturing" while some use shock fitting either with or without the usual techniques in other parts of the flow field. The basic problem with the former method is that it smears out shocks over several mesh points; the resulting distortion near the front is very undesirable in itself and can easily lead to further errors in interaction problems. On the other hand, shock fitting would be ideal for the problems at hand if it could be implemented in a straightforward fashion. However, the necessary theory is not available to track multidimensional discontinuities (in general, curved) and their interactions and transitions. Even if this were feasible, it would be still necessary to anticipate discontinuity formation (either automatically or otherwise) and prepare special algorithms for each along with the interactions which occur; this can be a very ad hoc procedure.

The advantages of Glimm's method are clear; discontinuities are tracked without smearing, and the formation, interaction, and transition of discontinuities is computed automatically. This, at least, is the situation in one space dimension. The most important open question for the method is its utility in higher dimensional problems (this, of course, is also the case for most other numerical techniques in compressible fluid flow). The results of Chorin presented in the section on the fractional step approach are extremely promising in this regard; they indicate that the sharp resolution of discontinuities of the one-dimensional scheme is also a property of the two-dimensional fractional step approach.

We intend to apply the method to a variety of multidimensional blast wave situations beginning with the problem of a spherical or cylindrical charge at height of burst. This becomes axisymmetric (two-dimensional) at the moment of impact of the shock on the ground. This problem contains transitions from regular to Mach reflection in addition to the initial strong shock wave followed by a contact discontinuity. If the initial energy is high enough, there is a further transition to multiple Mach reflection--this is amply documented by experiment although not understood theoretically. In a closely related problem, we will study the two-dimensional plane ramp problem with an incident plane shock as

17. Sod, G. A., "A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws," J.C.P. 27, April 1978, p. 1-31.

initial data (this is the "shock tube" problem whereas Chorin studied the "wind tunnel" problem), see Figure 7. There is a large body of experimental data available for this situation and the resulting unsteady flow fields are similar to the axisymmetric charge at height of burst problem. For the final applications, it will be necessary to use realistic equations of state; the appendix indicates that this will be feasible. This will also be very important in the more difficult charge at height of burst above water problem. In addition to the physical phenomena listed above, this problem also contains refraction of discontinuities as well. An advantage of Glimm's method for this problem is that as the leading shock approaches the water, there is no reason for the time step to approach zero as is the case for any Lagrangian scheme used on the problem; as the ambient atmosphere gets squeezed between the shocked air and the water, the probability of a random number lying in that portion of the Riemann problem solution corresponding to ambient atmosphere will approach zero and eventually the ambient atmosphere must disappear. Since water is only slightly compressible, its equation of state will not even resemble that for a γ-law gas and the development of a Riemann problem solver for water will be important.

Other problems involving material interfaces include a charge impacting on a solid surface and the design of shaped charges (which include a metal liner inside the charge). On the one hand, these problems should be amenable to the approach embodied in Glimm's method and the advantages of the latter indicated above will be important. On the other hand, the method has not yet been tested in problems of this type involving nonreflecting material interfaces. In one space dimension, this type of complication is not a problem; indeed, Chorin [Reference 3] points out that interfaces can be tracked with the same accuracy (in the mean) as can be attained by the method for shocks. In higher dimensions, this property remains to be tested.

We note also that research is ongoing to apply Glimm's method to problems involving combustion and detonation. We have outlined this development in the preceding section. Thus, it may be possible in the near future to use the method to model the detonation wave from an explosive instead of simply using experimental data to set up initial conditions for a hydrodynamic shock wave.

Finally, it is possible to couple Glimm's method to other techniques in separate flow regions. Chorin studies the interaction of a laminar incompressible boundary layer with an inviscid compressible interior flow [Reference 18]. Of course, a completely different method is necessary for the boundary layer. Glimm and Marchesin [Reference 16] consider "random shock tracking." This adaptability of the method may prove useful in future applications.

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18. Chorin, A. J., "Vortex Sheet Approximation of Boundary Layers," J.C.P. 27, June 1978, p. 428-442

APPENDIX

THE GODUNOV ITERATION FOR SOLVING RIEMANN PROBLEMS

Figure 1 contains most of the notation which we shall use. In addition, set $\tau = 1/\rho$, ε = internal energy, and U = shock speed. Solving a Riemann problem means: given $\underline{u}_\ell = (\rho_\ell, u_\ell, p_\ell)$ and $\underline{u}_r = (\rho_r, u_r, p_r)$, calculate $p_*, u_*, \rho_{*\ell}, \rho_{*r}$, the slopes of the shock (or shocks), contact discontinuity, and the leading and trailing edges of the rarefaction fan (or fans), and the solution inside the fan. In the application to Glimm's method, it is never necessary to calculate the entire solution; one always finds p_* and u_* and certain other parts of the solution depending on the random number selected. In particular, it is never necessary to explicitly find the entire rarefaction fan.

The first step in the solution is the Godunov iteration for the determination of p_*. This will be presented in some detail for a general equation of state. Next, the explicit formulas in the iteration will be derived for a γ-law gas and the complete solution of the Riemann problem will be set down in this case. This material is taken from References 3, 6, 7, 18, and 19. Finally, the Godunov iteration will be repeated for more complicated equations of state which arise in the study of chemical explosives. It will follow from this analysis that Glimm's method can be extended in a straightforward fashion to problems involving general equations of state. The remaining question is the (numerical) efficiency of the method for complicated equations of state. This is not resolved here except for a few general comments. In particular, we do not address the possibility of replacing Godunov's iteration scheme (which is very efficient in the γ-law gas case) by some other scheme. It will be clear that the iteration for p_* is the bulk of the computing time in the application of Riemann problem solutions to Glimm's method. Thus, we will not repeat the derivation of the complete solution to the Riemann problem for these equations of state.

Given p_* and p_r, it follows that the $*$-state and the right state are separated by either a shock wave or a centered rarefaction fan according as $p_* \geq p_r$ or $p_* \leq p_r$, respectively (entropy condition). The same result holds for p_* and p_ℓ. So, in order to proceed in the iteration, we set down the governing equations for a shock wave and for an isentropic rarefaction wave. To begin, suppose that the equation of state is of the form

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19. Sod, G. A., "The Computer Implementation of Glimm's Method," Lawrence Livermore Laboratory Report UCID-17252, 1976

$$p = p(\rho, \epsilon), \tag{A1}$$

and that equation (A1) may be solved explicitly for the internal energy ϵ,

$$\epsilon = \epsilon(\rho, p). \tag{A2}$$

By using the thermodynamic identity

$$\left(\frac{\partial p}{\partial \tau}\right)_S = \left(\frac{\partial p}{\partial \tau}\right)_\epsilon - p\left(\frac{\partial p}{\partial \epsilon}\right)_\tau \tag{A3}$$

and the equation of state (A1) and (A2), the equation for an isentrope can always be put in the form

$$\frac{dp}{d\tau} = h(\tau, p) \tag{A4}$$

or

$$p = p(\tau) = \int^\tau h(\tau', p')d\tau'. \tag{A5}$$

The solution of (A5) may or may not be obtainable in closed form; in any event, this solution will contain a constant of integration which depends on the isentrope. Additionally, there are also Riemann invariants which are of the form

$$\ell(\rho) \pm u = \text{const.} \tag{A6}$$

where

$$\ell(\rho) = \int^\rho \frac{c'd\rho'}{\rho'} = \int^{p'} \frac{dp'}{\rho'c'}. \tag{A7}$$

The constants of integration in (A7) may always be taken to be zero for gases; see Reference 7. The constant in (A6) depends on the value of the entropy. The sound speed c may be determined from

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S; \tag{A8}$$

the right-hand side of equation (A8) is obtainable from equation (A3). Therefore, for the case of a rarefaction wave between the $*$-state and the right state,

$$\ell(\rho_r) - u_r = \ell(\rho_*) - u_* \tag{A9}$$

$$H(p_r, \rho_r) = H(p_*, \rho_*) \tag{A10}$$

where H is obtained from integration of equation (A5). Equations (A9) and (A10) follow because the rarefaction wave is isentropic. Now, consider the case of a shock wave between the $*$-state and the right. We have the Rankine-Hugoniot conditions which are completely independent of the equation of state:

$$\rho_r(u_r - U) = \rho_*(u_* - U) = -M \tag{A11}$$

$$\rho_r(u_r - U)^2 + p_r = \rho_*(u_* - U)^2 + p_* \tag{A12}$$

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$$\varepsilon_r - \varepsilon_* = ((\tau_* - \tau_r)/2)(p_* + p_r) \tag{A13}$$

From equations (A11) and (A12), it follows that

$$M^2 = - (p_* - p_r)/(\tau_* - \tau_r). \tag{A14}$$

The exact analogues of equations (A9) - (A14) hold for the waves between the left state and the ($*$)-state.

After an initial guess for p_* has been made, the Godunov iteration proceeds by defining

$$M_r = (p_r - p_*)/(u_r - u_*) \tag{A15}$$

$$M_\ell = - (p_\ell - p_*)/(u_\ell - u_*). \tag{A16}$$

Observe that equations (A15), (A16) imply

$$p_* = (u_\ell - u_r + p_r/M_r + p_\ell/M_\ell)/((1/M_r) + (1/M_\ell)). \tag{A17}$$

Given an iterate p_*^ν, the Godunov iteration updates M_r and M to obtain the next iterate $p_*^{\nu+1}$. This is repeated until convergence is reached. Convergence criteria, accelerated convergence techniques, and the optimal choice of the initial guess p_*^0 will not be discussed here; see Reference 6. The quantities M_r and M_ℓ cannot be used in the form given in (A15), (A16) for this process because they involve the unknown u_*. The main idea in the iteration is to rewrite M_r and M_ℓ as functions of (p_r, ρ_r, p_*), (p_ℓ, ρ_ℓ, p_*) respectively, using equations (A1) - (A14). This is now carried out for the wave connecting the $*$-state with the right state. The wave moving to the left may be handled similarly.

Suppose that this wave is a shock. From equation (A11), $u_r - u_* = - M(\tau_r - \tau_*)$ which implies that $M_r = M$ by equation (A14). This means that

$$M_r^2 = - (p_r - p_*)/(\tau_r - \tau_*). \tag{A18}$$

This eliminates u_* but introduces τ_* into the equation for M_r. However, we can obtain $\tau_* = \tau_*(p_r, \rho_r, p_*)$ by using the Rankine-Hugoniot condition (A13) after expressing $\varepsilon_r = \varepsilon_r(\rho_r, p_r)$, $\varepsilon_* = \varepsilon_*(\rho_*, p_*)$ through the equation of state (A2). This involves the solution of a nonlinear algebraic equation. We are assuming that the equation of state is such that this nonlinear equation has exactly one root which satisfies the governing physical laws. Inserting this expression into equation (A18), we have $M_r = M_r(p_r, p_*, \rho_r)$ as desired.

Now suppose that the right wave is an isentropic rarefaction. Then, equations (A15) and (A9) lead directly to

$$M_r = (p_r - p_*)/(\ell(\rho_r) - \ell(\rho_*)) \tag{A19}$$

It follows from equations (A2), (A7), and (A8) that $\ell(\rho_r)$ is a function of ρ_r and p_r, and that $\ell(\rho_*)$ is a function of ρ_* and p_*. The remaining task is to eliminate ρ_*. This may be accomplished by solving the isentropic law (A10) for ρ_* in terms

of $p_r, \rho_r,$ and p_*. Once again, this is a nonlinear algebraic equation and we make the same assumption about its solution as was made above. Having solved this equation, one sees that equation (A19) exhibits the quantity M_r in the appropriate form.

Specializing to the case of a γ-law gas, the equation of state is

$$\varepsilon = \frac{1}{\gamma-1} \frac{p}{\rho} . \qquad (A20)$$

Application of the thermodynamic identity (A3) leads to

$$dp/d\tau = -\gamma p/\tau \qquad (A21)$$

and the solution of this ordinary differential equation is

$$p\rho^{-\gamma} = A(S) \qquad (A22)$$

where $A(S)$ is a constant of integration (which depends on the entropy, S). This is the isentropic law, and equation (A10) takes the form

$$p_r \rho_r^{-\gamma} = p_* \rho_*^{-\gamma}. \qquad (A23)$$

Similarly, one uses equations (A3), (A8), and (A20) to show that the sound speed satisfies

$$c^2 = \gamma p/\rho. \qquad (A24)$$

By inserting equation (A24) into equation (A7) for $\ell(\rho)$ and using equation (A22), we see that equation (A9), which expresses the constancy of the right Riemann invariant across the right wave, may be put in the form

$$\frac{2c_r}{\gamma-1} - u_r = \frac{2c_*}{\gamma-1} - u_*. \qquad (A25)$$

Therefore, if the right wave is a rarefaction, equations (A15), (A24), and (A25) lead to

$$M_r = (p_r - p_*)/\frac{2\gamma^{\frac{1}{2}}}{\gamma-1} \left[\left(\frac{p_r}{\rho_r}\right)^{\frac{1}{2}} - \left(\frac{p_*}{\rho_*}\right)^{\frac{1}{2}} \right] . \qquad (A26)$$

Using the isentropic law (A23), it follows (after some algebra) that

$$M_r = (p_r \rho_r)^{\frac{1}{2}} \frac{\gamma-1}{2\gamma^{\frac{1}{2}}} \left(1 - \frac{p_*}{p_r}\right) / \left(1 - \left(\frac{p_*}{p_r}\right)^{\gamma-1/2\gamma}\right) . \qquad (A27)$$

Thus, the general case (A19) for M_r has reduced to a simple algebraic formula. Suppose now that the right wave is a shock. Using the equation of state (A20) and the Rankine-Hugoniot condition (A13), one obtains

$$\tau_* = \tau_r \left(\frac{p_r + \mu^2 p_*}{\mu^2 p_r + p_*} \right)$$

where $\mu^2 = \frac{\gamma-1}{\gamma+1}$. When substituted into equation (A18),

$$M_r = (p_r \rho_r)^{\frac{1}{2}} \left(\frac{\gamma-1}{2} + \frac{\gamma+1}{2} \frac{p_*}{p_r} \right)^{\frac{1}{2}}. \tag{A29}$$

Equation (A29) is of the required form. Once again, it is merely an algebraic expression for M_r. Collecting our results, we see that

$$M_r = (p_r \rho_r)^{\frac{1}{2}} \phi(p_*/p_r) \tag{A30}$$

where

$$\phi(\alpha) = \begin{cases} \left(\frac{\gamma+1}{2} \alpha + \frac{\gamma-1}{2} \right)^{\frac{1}{2}}, & \alpha \geq 1 \\ \\ \frac{\gamma-1}{2\gamma^{\frac{1}{2}}} \frac{1-\alpha}{1-\alpha^{\alpha-1/2\gamma}}, & \alpha \leq 1. \end{cases} \tag{A31}$$

Observe that ϕ is continuous at $\alpha = 1$; indeed $\phi(1) = \gamma^{\frac{1}{2}}$. Formula (A30) also holds for M_ℓ except that p_r is replaced everywhere by p_ℓ. Thus, formula (A30), (A31), and (A17) define the iteration for p_* in the case of a γ-law gas. The question of convergence will not be taken up here except to note that the iteration converges very quickly in practice. Reference 6 may be consulted for further details.

The remainder of the solution to the Riemann problem is now derived. The equations will be for a γ-law gas, but the procedure easily generalizes to an arbitrary equation of state. One notes immediately that

$$u_* = (p_\ell - p_r + M_r u_r + M_\ell u_\ell)/(M_r + M_\ell) \tag{A32}$$

upon elimination of p_* from (A15), (A16). Observe that the equation for the slip line is $dx/dt = u_*$. Suppose there is a shock wave on the right (i.e., $p_* > p_r$). Equations (A11) gives us the shock speed U and the density to the right of the slip line ρ_{*r} since the quantity M is known from the iteration for ρ_*. A left-facing shock is handled similarly. The remaining case is a rarefaction. Suppose it is the right wave. The rarefaction is then bounded on the right by the line $dx/dt = u_r + c_r = u_r + (\gamma p_r \tau_r)^{\frac{1}{2}}$ and so this line is known a priori. It is bounded on the left by the line $dx/dt = u_* + c_* = u_* + (\gamma p_* \tau_*)^{\frac{1}{2}}$. The quantity τ_* (i.e., $1/\rho_{*r}$) is not known but can be found immediately from equation (A25). Thus, the left slope and the density ρ_{*r} are solved for simultaneously. If the solution is desired at some point p inside the rarefaction, one equates the slope of the characteristic $dx/dt = u+c$ to the slope of the line connecting p and the origin.

This is correct because the rarefaction fan is centered at the origin. Hence, $u+c = \beta$ where β is the slope. Also, the constancy of the right Riemann invariant and the isentropic law are available to give two more equations for the unknowns p, ρ, u. Hence, there is a total of three equations in three unknowns and the solution can be found. Observe that knowledge of the *-state is not necessary to find the solution inside the fan after one knows what the fan boundaries are (which does require information from the *-state).

In summary, it has been shown that the general solution of the Riemann problem can be reduced to simple algebraic formulas in the case of a γ-law gas. For applications to blast wave problems, there will be many situations when a γ-law approximation to the equation of state will be inappropriate either for the detonation products, the ambient atmosphere (e.g., after radiant heating has raised the temperature to a very high level), or some other media in the problem (e.g., in a blast above water, the water should be treated as a compressible medium if the blast strength is great enough). Hence, it is important to be able to efficiently solve Riemann problems for more complex equations of state in order to be able to apply Glimm's method to such problems. In addition, it is necessary to be able to solve Riemann problems across a material interface (e.g., the detonation product-ambient atmosphere interface). In this situation, the left and right states will have different equations of state.

Ritter [Reference 20] considers an explosive-metal system. For the compressible metal plate, Murnaghan's equation of state is used--

$$\varepsilon = (\nu-1)^{-1}[(p+\nu b)\tau - \nu b \tau_0] \tag{A33}$$

where $\nu > 1$, b, τ_0 are constants. The isentropic law is given by $p = b[(\frac{\rho}{\rho_0})^\nu - 1]$ and $c^2 = \nu(p+b)\tau$ gives the sound speed. An analysis similar to that leading up to equation (A36) shows that for the equation of state (A33), the quantity M_r is given by

$$M_r = [(p_r + b)\rho_r]^{\frac{1}{2}}\phi((p_*+b)/(p_r + b)) \tag{A34}$$

where

$$\phi(\alpha) = \begin{cases} (\frac{\nu+1}{2}\alpha + \frac{\nu-1}{2})^{\frac{1}{2}} & , \quad \alpha \geq 1 \\ \dfrac{\nu-1}{2\nu}\dfrac{1-\alpha}{1-\alpha^{(\nu-1)/2\nu}} & , \quad \alpha \leq 1 \end{cases} \tag{A35}$$

Hence, this equation of state is no more difficult to work with than that of a γ-law gas. Glimm's method for this problem is compared with previous experimental results in Reference 20.

20. Ritter, Z. W., "A New Method for Calculating Hydrodynamic Behavior of Plane One-Dimensional Explosive-Metal Systems," 1977, preprint

Kury, et al [Reference 21] use the equation of state

$$p = A(1 - \frac{\omega}{R_1 \tau})e^{-R_1 \tau} + B(1 - \frac{\omega}{R_2 \tau})e^{-R_2 \tau} + \frac{\omega \varepsilon}{\tau} \tag{A36}$$

for a chemical explosive known as Comp B, Grade A. In equation (A36), the quantities A, B, R_1, R_2, ω are constants. Applying equation (A3), the following ordinary differential equation is obtained for the isentrope (after some algebraic manipulations using the equation of state (A36)):

$$\frac{dp}{d\tau} + \frac{1+\omega}{\tau} p = A(\frac{\omega+1}{\tau} - R_1)e^{-R_1 \tau} + B(\frac{\omega+1}{\tau} - R_2)e^{-R_2 \tau}. \tag{A37}$$

The integrating factor for equation (A37) is $\tau^{(\omega+1)}$ and one obtains after a straightforward calculation that the solution of (A37) is

$$p = Ae^{-R_1 \tau} + Be^{-R_2 \tau} + C\tau^{-(\omega+1)}. \tag{A38}$$

Here, C is a constant of integration which plays the same role as the constant $A(S)$ in equation (A22) for the γ-law gas isentropes; that is, C is different for different isentropes. Combining equations (A8) and (A37), it is easy to see that the sound speed is given by

$$c^2 = (1+\omega)p\tau - A\tau(1+\omega - R_1\tau)e^{-R_1 \tau} - B\tau(1+\omega - R_2\tau)e^{-R_2 \tau} ; \tag{A39}$$

substituting the expression (A38) for the pressure p in equation (A39), one obtains

$$c^2 = D\tau^{-\omega} + AR_1\tau^2 e^{-R_1 \tau} + BR_2\tau^2 e^{-R_2 \tau} \tag{A40}$$

where D is another constant depending on the isentrope. Equation (A7) for the quantity $\ell(\rho)$ appearing in the definition of Riemann invariant becomes

$$\ell(\rho') = \int^{\rho'} (D\rho^{\omega-2} + AR_1\rho^{-4}e^{-R_1/\rho} + BR_2\rho^{-4}e^{-R_2/\rho})^{1/2}d\rho. \tag{A41}$$

(Note: the constant ω is always greater than zero so that $\ell(\rho)$ does not blow up near $\rho = 0$. Indeed, according to Courant and Friedrichs [Reference 7], one may take $\ell(\rho) = 0$ for $\rho = 0$, at least for gases.) The analogue of equation (A10) for the isentropic law (A38) is

$$\tau_r^{\omega+1}(p_r - Ae^{-R_1 \tau_r} - Be^{-R_2 \tau_r}) = \tau_*^{\omega+1}(p_* - Ae^{-R_1 \tau_*} - Be^{-R_2 \tau_*}). \tag{A42}$$

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21. Kury, J. W., et.al., "Metal Acceleration by Chemical Explosions," Fourth Symposium (Int.) on Detonation, Oct. 1965, p. 3-13.

Hence, if the right wave is a centered rarefaction, M_r is found from equation (A19) which, in view of equation (A41), takes the form

$$M_r = (p_r - p_*) / \int_{\rho_r}^{\rho_*(p_r, \rho_r, p_*)} (D\rho^{\omega-2} + AR_1\rho^{-4}e^{-R_1/\rho} + BR_2\rho^{-4}e^{-R_2/\rho})^{\frac{1}{2}}d\rho \quad (A43)$$

where $\rho_* = \rho_*(p_r, \rho_r, p_*)$ is the solution of equation (A42). On the other hand, if the right wave is a shock, M_r is found from equation (A18),

$$M_r^2 = (p_r - p_*)/(\tau_r - \tau_*(p_r, \rho_r, p_*)), \quad (A44)$$

where τ_* is the solution of the Rankine-Hugoniot condition (A13); the quantities $\varepsilon_r, \varepsilon_*$ are easily eliminated in favor of p_r, ρ_r and p_*, ρ_* by direct use of the equation of state (A36) and the result is a nonlinear algebraic equation for τ_* in terms of p_r, ρ_r, and p_*. Similar results hold for the left wave. In summary, solving the Riemann problem for Comp B, Grade A will involve the solution of a nonlinear algebraic equation, either (A42) or (A13), and possibly a quadrature (equation A(43)), at each step in the Godunov iteration.

Thus, Comp B, Grade A is an example of an equation of state for which nontrivial numerical procedures must be introduced at each step in the iteration. However, it does not represent the worst possible case—in general, the differential equation for the isentrope (A4) will not be reducible to quadratures and it will be necessary to use a numerical ordinary differential equation solver in order to obtain the isentropic law. The situation for the equation of state of water illustrates this eventuality. Among other possibilities, we will consider the γ-law type and the Sternberg-Walker equations of state which are both presented in Reference 22. The former is given by

$$[p_0\tau_0 + \gamma A(\tau_0-\tau) + (\gamma-1)\varepsilon](\gamma-1)\varepsilon/\tau \quad (A45)$$

where γ, A, p_0, τ_0 are constants. A computation shows that the isentropic law (A4) takes the form

$$\frac{dp}{d\tau} + [(\gamma-1)(p_0\tau_0 + \gamma A(\tau_0-\tau)) + 1]p/\tau$$

$$= \frac{1}{2}(p_0\tau_0/\tau^2 - \gamma A/\tau)[-(p_0\tau_0 + \gamma A(\tau_0-\tau)) \pm \sqrt{(p_0\tau_0+\gamma A(\tau_0-\tau))^2+4p\tau} \,]. \quad (A46)$$

The square root arises because the internal energy, ε, must be eliminated using the equation of state (A45) which exhibits an ε^2 term in the equation for p. Thus, in an iteration of the Godunov method which contains a rarefaction as the right wave, the only way to find the value of τ_* corresponding to p_r, τ_r and the updated p_* is to numerically solve the ordinary differential equation (A46) using

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22. Enig, J. W., "The Unsteady Regular and Mach Reflection Resulting from the Interaction of Spherical Explosion Shock Waves in Water," <u>Sixth Symposium (Int.) on Detonation</u> 1976, p. 570-590.

$p = p_r$ at $\tau = \tau_r$ as the initial data until $p = p_*$ is reached at which point the integration is stopped and τ_* is set equal to the current value of τ. Also, one can proceed as in the previous example and obtain the sound speed as a function of p and τ by combining equations (A8) and (A46); unfortunately, equation (A46) must be used to eliminate p in terms of τ which means that the equation for the Riemann invariant will involve the solution of an ordinary differential equation instead of a simple quadrature as was the case in equation (A41) for Comp B, Grade A. On the other hand, the case of a shock wave will differ little in complexity from the situation for Comp B, Grade A. The Sternberg-Walker equation is

$$p = f_1(\varepsilon)/\tau + f_2(\varepsilon)/\tau^3 + f_3(\varepsilon)/\tau^5 + f_4(\varepsilon)/\tau^7 \tag{A47}$$

where the f_i, $i=1,\cdots,4$ are polynomials. In this case, it may even be necessary to use numerical methods to eliminate ε in order to obtain the differential equation for the isentrope. In any event, unless the choice of the polynomials f_i are extremely fortuitous, the analogue of equation (A46) will be far more complicated. Although we could set down some more equations, the foregoing brief analysis should indicate the problems to be solved in the case of a general equation of state.

One factor left to be considered is that an equation of state is sometimes available only in the form of a graph or table. This may be the case for accurate computations involving real air at extremely high temperatures. Two approaches are available in this event for constructing solutions to the Riemann problem. First, the data can be fitted by an equation of state in functional form and the problem can be solved as before. Second the data can be preprocessed into a format usable for table lookup in the various steps of the Godunov iteration. Parenthetically, we note that this option is also available for a very complicated functional equation of state. The choice between these two alternatives is clearly a matter of computational efficiency.

Finally, we point out that the Riemann problem is not necessarily well-posed for an arbitrary equation of state. This is an area of current interest in the mathematical literature. For example, recall that in the foregoing we have occasionally imposed various conditions such as unique solvability of nonlinear equations arising from the equation of state (unique in the physical sense, not mathematically unique--that is the solution must satisfy the equation and satisfy entropy and possibly other compatibility conditions). It will be of interest to obtain theorems showing the Riemann problem is well-posed for the equations of state that are used in blast wave theory. It might be of even greater interest to obtain negative results. Since the Riemann problem is a reasonable initial configuration, this would show that the equation of state is unphysical.

CONCLUSIONS

We have shown in this report that Glimm's method is applicable to a variety of problems involving strong shocks, material interfaces, and blast waves. A priori, the method has significant advantages over currently available finite difference techniques for problems of this type. Whether or not these advantages outweigh disadvantages must await further numerical tests and possibly theoretical algorithm development, especially in multidimensional situations. The last section outlines the directions that we would like to see this research take. The numerical tests reported here using strongly nonuniform waves are very promising and we expect similar results for two-dimensional analogues. The appendix shows that the extension of the method to a general equation of state is straightforward, but (potentially) very tedious from the point of view of developing an efficient computer code. In summary, there is an excellent chance that Glimm's method will, in the near future, be able to yield qualitatively superior results and thereby advance the field of numerical analysis of blast wave effects.

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